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Note

Limits of Chebyshev Polynomials When the Argument Is a Ratio of Cosines

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Two new limits involving Chebyshev polynomials of the first and second kinds are given. These limits are useful in certain engineering applications. The proofs are based on the Mehler–Heine theorem for Jacobi polynomials.

Let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials. It is evident from the representation [1, (4.21.2)]

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n!} \sum_{\nu=0}^{n} \binom{n}{\nu} (n+\alpha+\beta+1)_\nu (\alpha+\nu+1)_{n-\nu} \left(\frac{x-1}{2}\right)^\nu \quad (1)$$

that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n in x and in the parameters α and β. Hence, $P_n^{(\alpha,\beta)}(x)$ can be extended to all complex values of α, β, and x. In this note, α and β are restricted to be real numbers.

For any complex number x, the Mehler–Heine theorem [1, Theorem 8.1.1] states that

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{x}{n}\right) = (x/2)^{-\alpha} J_\alpha(x), \quad (2)$$

where $J_\alpha(x)$ is the Bessel function of the first kind of order α [1, (1.71.1)]. Szegö's proof of (2) actually establishes that

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{x^2}{2n^2} + o(n^{-2})\right) = (x/2)^{-\alpha} J_\alpha(x).$$

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Consequently, for all complex x and y,

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(\frac{\cos(x/n)}{\cos(y/n)}\right)=\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{x^2-y^2}{2n^2}+o(n^{-2})\right)$$
$$=\left(\frac{1}{2}\sqrt{x^2-y^2}\right)^{-\alpha}J_\alpha(\sqrt{x^2-y^2}). \tag{3}$$

Like the Mehler–Heine theorem, this result holds uniformly for x and y in every bounded region of the complex plane.

The limit (3) has interesting special forms for the Chebyshev polynomials $T_n(z)$ and $U_n(z)$ of the first and second kinds, respectively. Substituting the identities

$$P_n^{(-1/2,-1/2)}(z)=\frac{(2n)!}{2^{2n}(n!)^2}T_n(z), \qquad n\geqslant 1,$$

and

$$J_{-1/2}(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos z$$

in (3) and applying Stirling's formula gives

$$\lim_{n\to\infty} T_n\left(\frac{\cos(x/n)}{\cos(y/n)}\right)=\cos\sqrt{x^2-y^2}. \tag{4}$$

Similarly, substituting

$$P_n^{(1/2,1/2)}(z)=\frac{(2n+2)!}{2^{2n+1}((n+1)!)^2}U_n(z), \qquad n\geqslant 0,$$

and

$$J_{1/2}(z)=\left(\frac{2}{\pi z}\right)^{1/2}\sin z$$

in (3) and applying Stirling's formula gives

$$\lim_{n\to\infty} n^{-1}U_n\left(\frac{\cos(x/n)}{\cos(y/n)}\right)=\frac{\sin\sqrt{x^2-y^2}}{\sqrt{x^2-y^2}}. \tag{5}$$

These limiting forms do not seem to be mentioned elsewhere in the literature.

A result similar to (4) is used implicitly in an antenna design application [2]. The result (5) is shown in [3] to be intimately related to the so-called

Kaiser–Bessel window in digital filter design. These applications require knowledge of the cosine transform of the right-hand side of (3), which is provided by a special case of Sonine's second finite integral [4, p. 376] for $\alpha > -\frac{1}{2}$. In particular,

$$\lim_{n\to\infty} n^{-1}U_n\left(\frac{\cos(x/n)}{\cos(y/n)}\right) = \int_0^1 I_0(y\sqrt{1-\zeta^2})\cos x\zeta\, d\zeta, \tag{6}$$

where $I_\nu(z)$ is the modified Bessel function of order ν. Sonine's second finite integral diverges for $\alpha = -\frac{1}{2}$; however, the cosine transform of $\cos((x^2-y^2)^{1/2})$ is known [5, (871.2)], so that

$$\lim_{n\to\infty} T_n\left(\frac{\cos(x/n)}{\cos(y/n)}\right) = \cos x + y\int_0^1 \frac{I_1(y\sqrt{1-\zeta^2})}{\sqrt{1-\zeta^2}}\cos x\zeta\, d\zeta. \tag{7}$$

It is evident from (6) and (7) that the limiting forms have finite support (i.e., are bandlimited) and, thus, are of exponential type.

An extremal property of $\cos((x^2-y^2)^{1/2})$ in the space of functions of exponential type is given in [6]. The proof is based on a theorem in [7]. Whether or not the limit function (3) has extremal properties in this space is not known to the author.

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